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Divisorially graded coalgebras☆

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Abstract

In this paper, we classify torsion theories in the category of graded comodules over a graded coalgebra. Moreover, we give a structure theorem for divisorially graded coalgebras in terms of Picard groups. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [8] a systematic study of torsion theories in the category of comodules over a coalgebra was done. It seems to be interesting to consider torsion theories in the category of graded comodules over a graded coalgebra. In the literature on torsion theories for graded modules we find, as one of the most relevant results, the Relative Dade's Theorem which deals with the connection between some important categories associated to a graded ring. Also, divisorially graded rings are important generalizations of strongly graded rings (this class of graded rings includes crossed-products), and are studied in [6].

Our goal in this paper is, in a first step, to classify torsion theories in the category of right graded comodules over a graded coalgebra C, writing this category as gr^C. Part of the work uses the non-graded methods (see [8]), but immediately we will see that the graded theory changes, as in the graded ring theory, when we want to analyse torsion theories induced from non-graded subcoalgebras and those induced from graded subcoalgebras of C. In this way, the concept of *rigid torsion theory* appears, and is used in the relative Dade's theorem for graded coalgebras, Theorem 4.1. Then we

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introduce the concept of divisorially G-graded coalgebra. This kind of coalgebras have the important property that the group G has to be finite, Corollary 4.2 (note that divisorially graded rings do have not this property). We finish the paper by giving a structure theorem for strongly graded coalgebras and divisorially graded coalgebras in terms of Picard groups, in Section 5.

2. Notation and preliminaries

Let k be a field. Vector spaces over k are called k-spaces, and linear maps between k-spaces are called k-maps. A coalgebra over k is a k-space C together with two k-linear maps $\Delta: C \to C\otimes C$ (the unadorned tensor product is understood to be over k) and $\varepsilon: C\to k$ such that $(1\otimes\Delta)\Delta = (\Delta\otimes 1)\Delta$ and $(1\otimes\varepsilon)\Delta = (\varepsilon\otimes 1)\Delta = 1$. We shall use the so-called “sigma notation” (see Sweedler’s book [11] or Abe’s book [1]), that is $\Delta(c) = \sum_{(c)} c_1\otimes c_2$ if $c\in C$.

If C is a coalgebra, a right C-comodule is a k-space M with a k-map $\rho_M : M \to M\otimes C$ such that $(\rho_M\otimes 1)\rho_M = (1\otimes\Delta)\rho_M$ and $(1\otimes\varepsilon)\rho_M = 1$. We also use the sigma notation for C-comodules, i.e., $\rho(m) = \sum_{(m)} m_0\otimes m_1$, $m_0\in M, m_1\in C$. If M and N are C-comodules, a comodule map from M to N is a k-map $f: M\to N$ such that $(f\otimes 1)\rho_M = \rho_N f$. The k-space of all comodules maps from M to N written as $Com_C(M,N)$ and $\mathscr{M}^C$ denotes the category of right C-comodules. In the same way, we can construct the category ${}^C\mathscr{M}$ of left C-comodules.

If C and D are two coalgebras, a (C,D)-bicomodule is a left C-comodule and a right D-comodule M, such that the C-comodule structure map $\rho_M^- : M\to C\otimes M$ is a D-comodule map, or equivalently the D-comodule structure map $\rho_M^+ : M\to M\otimes D$ is a C-comodule map.

It is well known that $\mathscr{M}^C$ is an abelian category (see [11, 1], etc.). In fact, $\mathscr{M}^C$ is a Grothendieck category.

Let G be a group with $e\in G$ the identity element of G. Following [7], a coalgebra C is called G-graded coalgebra if C admits a decomposition as a direct sum of k-spaces $C = \bigoplus_{\sigma\in G} C_\sigma$ such that

(i) $\Delta(C_\sigma)\subseteq \sum_{\lambda\mu=\sigma} C_\lambda\otimes C_\mu$ for any $\sigma\in G$;

(ii) $\varepsilon(C_\sigma) = 0$ for any $\sigma\neq e$.

If M is a right C-comodule then M is called a G-graded comodule over C if M admits a decomposition as a direct sum of k-spaces $M = \bigoplus_{\sigma\in G} M_\sigma$ such that $\rho_M(M_\sigma)\subseteq\sum_{\lambda\mu=\sigma} M_\lambda\otimes C_\mu$ for any $\sigma\in G$. For any element $m\in M$ we have the decomposition $m = \sum_{\sigma\in G} m_\sigma$, $m_\sigma\in M_\sigma$ (the sum has only a finite number of non-zero elements). The non-zero elements m_σ, $\sigma\in G$ are called the homogeneous components of m; m_σ is called the homogeneous element of degree σ and we write $deg(m_\sigma) = \sigma$.

Associated to any G-graded coalgebra $C = \bigoplus_{\sigma\in G} C_\sigma$ we have the category gr^C of all right graded C-comodules. In this category if $M = \bigoplus_{\sigma\in G} M_\sigma$ and $N = \bigoplus_{\sigma\in G} N_\sigma$ are two objects, then the morphisms from M to N is the set

$$Com_{gr^C}(M,N) = \{f\in Com_C(M,N) \mid f(M_\sigma)\subseteq N_\sigma \text{ for all } \sigma\in G\}.$$

It is easy to verify that gr^C is an abelian category (in fact gr^C is also a Grothendieck category, cf. [7, Section 4]). Analogously, we can define ${}^Cgr$ the category of all left G-graded C-comodules.

If $M = \bigoplus_{\sigma\in G} M_\sigma$ is a graded right C-comodule for any $\sigma \in G$ we write $\pi_\sigma^M : M \to M_\sigma$ as the canonical projection.

The following facts appear in [7, Section 3]. They will be used very frequently in this paper.

Let $M = \bigoplus_{\sigma\in G} M_\sigma$ be a right G-graded C-comodule.

(1) If $\sigma, \tau \in G$ there exists a unique k-morphism $u_{\sigma,\tau}^M : M_{\sigma\tau} \to M_\sigma \otimes C_\tau$ such that $u_{\sigma,\tau}^M \circ \pi_{\sigma\tau}^M = (\pi_\sigma^M \otimes \pi_\tau^C) \circ \rho_M$.

(2) For any σ, τ, $\lambda \in G$: $(u_{\sigma,\tau}^M \otimes 1) \circ u_{\sigma\tau,\lambda}^M = (1 \otimes u_{\tau,\lambda}^C) \circ u_{\sigma,\tau\lambda}^M$.

(3) If $\sigma \in G$, $(1 \otimes \varepsilon) \circ u_{\sigma,e}^M = 1$.

(4) If we write $\Delta_e = u_{e,e}^C : C_e \to C_e \otimes C_e$, then $(C_e, \Delta_e, \varepsilon)$ is a coalgebra and $\pi_e : C \to C_e$ is a morphism of coalgebras. Moreover, if $M = \bigoplus_{\sigma\in G} M_\sigma$ is a right C-comodule, then for any $\sigma \in G$, M_σ is a right C_e-comodule via the canonical map $u_{\sigma,e}^M : M_\sigma \to M_\sigma \otimes C_e$.

(5) Let $M = \bigoplus_{\sigma\in G} M_\sigma$, $N = \bigoplus_{\sigma\in G} N_\sigma$ be two objects in gr^C. Let $f \in Com_{gr^C}(M, N)$. Since $f(M_\sigma) \subseteq N_\sigma$ for any $\sigma \in G$, we write $f_\sigma : M_\sigma \to N_\sigma$ as the restriction of f, for $\sigma \in G$. Then, f_σ is a morphism in the category $\mathscr{M}^{C_e}$.

Let $M = \bigoplus_{\sigma\in G} M_\sigma$ be an object in gr^C and $\sigma \in G$. As for graded modules, we can define the σ-suspension of M, $M(\sigma)$, which is again an object in gr^C G-graded by $M(\sigma)_\lambda = M_{\sigma\lambda}$ for any $\lambda \in G$. It is clear that $M \mapsto M(\sigma)$ defines an isomorphism of categories from gr^C to gr^C.

We write $U : gr^C \to \mathscr{M}^C$ as the forgetful functor. U is an exact functor and it has an exact right adjoint functor $F : \mathscr{M}^C \to gr^C$. Moreover, if the group G is finite, then F is also a left adjoint of U (cf. [7, Proposition 3.4]).

Let C be an arbitrary coalgebra. If M is a right C-comodule and N is a left C-comodule, the *cotensor product* $M \,\square_C\, N$ is the kernel of the k-map $\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \to M \otimes C \otimes N$. Following [3], the cotensor product is a left exact functor $\mathscr{M}^C \times {}^C\mathscr{M} \to M_k$ (M_k is the category of k-spaces). Moreover, the mapping $m \otimes c \mapsto \varepsilon(c)m$ and $c \otimes n \mapsto \varepsilon(c)n$ yield natural isomorphism $M \,\square_C\, C \cong M$ and $C \,\square_C\, N \cong N$.

Now we consider the graded case. Let C be a G-graded coalgebra. We know that C_e is a coalgebra and the canonical map $\pi_e : C \to C_e$ is a morphism of coalgebras. Let $M \in \mathscr{M}^{C_e}$ be a right C_e-comodule. Since C has in the natural way (via the morphism π_e) a structure of (C_e, C)-bicomodule we can consider $M \,\square_{C_e}\, C \in \mathscr{M}^C$. Since $C = \bigoplus_{\sigma\in G} C_\sigma$ and C_σ is a (C_e, C_e)-bicomodule, for any $\sigma \in G$, $M \,\square_{C_e}\, C = \bigoplus_{\sigma\in G}(M \,\square_{C_e}\, C_\sigma)$. Thus, $M \,\square_{C_e}\, C$ has the natural graduation if we put $(M \,\square_{C_e}\, C)_\sigma = M \,\square_{C_e}\, C_\sigma$. It is proved that $M \,\square_{C_e}\, C$ is a right G-graded C-comodule.

Clearly, in this way, we obtain the functor $- \,\square_{C_e}\, C : \mathscr{M}^{C_e} \to gr^C$, $M \mapsto M \,\square_{C_e}\, C$, $M \in \mathscr{M}^{C_e}$. This functor is called induced functor and is left exact.

On the other hand, we have the functor $(-)_e : gr^C \to \mathscr{M}^{C_e}$ defined by $M \mapsto M_e$, for any $M \in gr^C$. This functor is exact. More general for any $\sigma \in G$ we have the exact functor $(-)_\sigma : gr^C \to \mathscr{M}^{C_e}$, defined by $M \mapsto M_\sigma$, where $M = \bigoplus_{\lambda\in G} M_\lambda$.

By [7, Proposition 4.1], the functor $-\square_{C_e} C\colon \mathscr{M}^{C_e} \to gr^C$ is a right adjoint of the functor $(-)_e\colon gr^C \to \mathscr{M}^{C_e}$. Moreover, we have the composition $(-)_e \circ (-\square_{C_e} C) \cong 1_{\mathscr{M}^{C_e}}$.

Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a G-graded coalgebra. Following [1] or [11], $C^* = Hom_k(C,k)$ has a natural structure of ring. Indeed if $f,g \in C^*$, then $fg = (f\otimes g)\Delta$ (here $k\otimes k \simeq k$). Hence, if $c \in C$ and $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$ then $(fg)(c) = \sum_{(c)} f(c_1)g(c_2)$. Let $\rho_M\colon M \to M\otimes C$ be a right C-comodule. Then M has a natural structure of left C^*-module.

For any $\sigma \in G$ we put $R_\sigma = \{f \in C^* \mid f(C_\tau) = 0 \text{ for all } \tau \neq \sigma\}$ (note that $R_\sigma \cong C_\sigma^*$ as vector spaces). We define $R = \sum_{\sigma\in G} R_\sigma = \bigoplus_{\sigma\in G} R_\sigma$. By [7, Proposition 6.1], R is a G-graded ring with $\varepsilon\colon C \to k$ the identity map. Actually, R is the graded dual algebra of the graded coalgebra C.

Let $V = \bigoplus_{\sigma\in G} V_g$ be a G-graded vector space. Then $V' = \bigoplus_{\sigma\in G} V_g^*$ is the graded dual of V. Then for any graded subspace W of V, the graded annihilating space $W^{gr\perp} = \{f \in V' \mid f(W) = 0\}$ of W is dense in $W^\perp$, i.e., $(W^{gr\perp})^\perp = (W^\perp)^\perp = W$. This fact will be used in the next section.

3. Torsion theories on gr^C

Let $\mathscr{A}$ be a Grothendieck category and $\mathscr{C}$ a full subcategory of $\mathscr{A}$. $\mathscr{C}$ is called *closed* [4, p. 395], if $\mathscr{C}$ is closed under subobjects, quotient objects and direct sums. If $\mathscr{C}$ is furthermore closed under extensions, then $\mathscr{C}$ is called a *localizing subcategory* of $\mathscr{A}$. It may be easily seen that a closed subcategory of a Grothendieck category is also a Grothendieck category (indeed, if $U \in \mathscr{A}$ is the generator, then $\{U/K \mid K \subset U$ is a subobject in $\mathscr{A}$ of U and $U/K \in \mathscr{C}\}$ is a family of generators of $\mathscr{C}$ and the direct sum is a generator).

If $\mathscr{C}$ is closed, the sum of all the subobjects, $t_{\mathscr{C}}(M)$, of $M \in \mathscr{A}$ which belong to $\mathscr{C}$ defines a left exact subfunctor $t_{\mathscr{C}}\colon \mathscr{A} \to \mathscr{A}$ of the identity functor of $\mathscr{A}$. This functor is called the preradical functor associated to $\mathscr{C}$. If $M \in \mathscr{A}$ and $M = t_{\mathscr{C}}(M)$ (resp. $0 = t_{\mathscr{C}}(M)$), then M is said to be $\mathscr{C}$-torsion (resp. $\mathscr{C}$-torsion free) object.

If $\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$, then $t_{\mathscr{C}}(M/t_{\mathscr{C}}(M)) = 0$. Therefore, in this case $t_{\mathscr{C}}$ is a radical. Following [10, Ch. VI], a closed subcategory of $\mathscr{A}$ is an *hereditary pretorsion theory* in $\mathscr{A}$ and a localizing subcategory of $\mathscr{A}$ is an *hereditary torsion theory* in $\mathscr{A}$.

Since the category gr^C is a Grothendieck category, the aim of this section is to study the hereditary pretorsion (resp. torsion) theories on gr^C.

Definition 3.1. Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a G-graded coalgebra.

(1) A localizing subcategory $\mathscr{C}$ in $\mathscr{M}^C$ will be called graded if $t_{\mathscr{C}}(M) \in gr^C$ for all $M \in gr^C$.

(2) A localizing subcategory $\mathscr{C}$ in gr^C will be called rigid if $M(\sigma) \in \mathscr{C}$ for all $M \in \mathscr{C}$ and $\sigma \in G$.

(3) A localizing subcategory $\mathscr{T}$ in $\mathscr{M}^{C_e}$ will be called G-stable if for all $T \in \mathscr{T}$, $T \,\square_{C_e}\, C_\sigma \in \mathscr{T}$ for all $\sigma \in G$.

Remark. It is clear that a graded localizing subcategory $\mathscr{C}$ in $\mathscr{M}^C$ induces one in gr^C, which we will write $\mathscr{C}^{gr}$. It is straightforward that $\mathscr{C}^{gr}$ is rigid.

Lemma 3.2. *Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a cocommutative G-graded coalgebra. Then, any closed subcategory in $\mathscr{M}^{C_e}$ is G-stable.*

Proof. Let $\mathscr{T}$ be a closed subcategory in $\mathscr{M}^{C_e}$. Let $T \in \mathscr{T}$ and $\sigma \in G$. Since $C_\sigma \in {}^{C_e}\mathscr{M}^{C_e}$, we have a monomorphism $C_\sigma \to C_e^{(I)}$ in ${}^{C_e}\mathscr{M}$ for some index set I. Then $T \,\square_{C_e}\, C_\sigma \to T \,\square_{C_e}\, C_e^{(I)}$ is a monomorphism in $\mathscr{M}^{C_e}$. But, $T \,\square_{C_e}\, C_e^{(I)} \cong T^{(I)}$ and so $T \,\square_{C_e}\, C_\sigma \in \mathscr{T}$. $\square$

Remark. As a consequence of Lemma 3.2, any localizing subcategory in $\mathscr{M}^{C_e}$ induces a localizing subcategory in gr^C.

Recall that a G-graded coalgebra, $C = \bigoplus_{\sigma\in G} C_\sigma$ is said to be *strongly graded coalgebra* if for $\sigma, \tau \in G$ the canonical morphisms $u^C_{\sigma,\tau}: C_{\sigma\tau} \to C_\sigma \otimes C_\tau$, $c \mapsto \sum \pi_\sigma(c_1) \otimes \pi_\tau(c_2)$ are monomorphisms (see [7]). In the next theorem, we collect some equivalent conditions for strongly graded coalgebras.

Theorem 3.3. *Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a G-graded coalgebra. The following conditions are equivalent*:

(a) *C is strongly graded.*

(b) *The induction functor $- \,\square_{C_e}\, C: \mathscr{M}^{C_e} \to gr^C$ is an equivalence of categories* (*cf.* [7, *Theorem* 5.4]).

(c) *The functor $(-)_e: gr^C \to \mathscr{M}^{C_e}$ is an equivalence of categories* (*cf.* [7, *Theorem* 5.4]).

(d) *For some $\sigma \in G$, $(gr^C)_\sigma = \{0\}$, where $(gr^C)_\sigma = \{M = \bigoplus_{\lambda\in G} M_\lambda \in gr^C \mid M_\sigma = \{0\}\}$* (*cf.* [9, *Proposition* 3.8]).

Now, we give a correspondence theorem between torsion theories in $\mathscr{M}^{C_e}$ and gr^C, when the graded coalgebra C is strongly graded.

Theorem 3.4. *Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a cocommutative strongly G-graded coalgebra. Then every localizing subcategory of gr^C is rigid and there is a one–one correspondence between localizing subcategories in $\mathscr{M}^{C_e}$ and localizing subcategories in gr^C.*

Proof. By Theorem 3.3, the $\mathscr{M}^{C_e}$ and gr^C are k-linear equivalent provided C is strongly graded. Hence, there is a one–one correspondence between their localizing subcategories.

Now, we prove that any localizing subcategory of gr^C is rigid. Let $\mathscr{D}$ be a localizing subcategory in gr^C. We consider the corresponding localizing subcategory in $\mathscr{M}^{C_e}$: $\mathscr{D}_{C_e} = \{N \in \mathscr{M}^{C_e} \mid N \,\square_{C_e}\, C \in \mathscr{D}\}$. By Lemma 3.2, $\mathscr{D}_{C_e}$ is G-stable. If $X \in \mathscr{D}$

then $X_e \in \mathscr{D}_{C_e}$. Hence, $X(\sigma)_e = X_\sigma \cong X_e \,\Box_{C_e}\, C_\sigma \in \mathscr{D}_{C_e}$ for all $\sigma \in G$. So $X(\sigma) \in \mathscr{D}$ for all $\sigma \in G$. $\Box$

The next results prove that rigid closed subcategories in gr^C are given by graded subcoalgebras of C. These results follow the same ideas that the ones for the non-graded case (cf. [8]). Also, we will prove that any rigid closed subcategory in gr^C is closed under direct products. Anyway, there exists non-rigid closed subcategories in gr^C which are closed under direct product. For example, $(gr^C)_\sigma$, for all $\sigma \in G$, have this property (see [9, Proposition 3.1]).

Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a graded coalgebra and let R be the graded associated ring. Let $M = \bigoplus_{\sigma\in G} M_\sigma$ be a graded right C-comodule. Recall, [7], that we can define a left graded R-module, $\overline{M}$, by putting $\overline{M} = M$ as k-spaces but for $\sigma \in G$, $\overline{M}_\sigma = M_{\sigma^{-1}}$. Also if $N \in R-gr$ is gr-rational, we can define a right graded C-comodule $\overline{N}$ in the following way: $\overline{N} = N$ as k-spaces; for $\sigma \in G$ we put $\overline{M}_\sigma = M_{\sigma^{-1}}$ and $\rho_M : \overline{M} \to \overline{M} \otimes C$ is the canonical map.

The following result will be useful in the sequel.

Lemma 3.5. *Let G be a group and let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a G-graded coalgebra. Let A be a subcoalgebra of C. If A is a G-graded linear subspace of C, then A is a G-graded subcoalgebra of C.*

Proof. Let $A_\sigma = C_\sigma \cap A$. Obviously, $\Delta(A_\sigma) \subseteq \bigoplus_{\tau\mu=\sigma} C_\tau \otimes C_\mu$. On the other hand, $\Delta(A_\sigma) \subseteq \bigoplus_{\tau,\mu\in G} A_\tau \otimes A_\mu$. Therefore,

$$\Delta(A_\sigma) \subseteq \left(\bigoplus_{\tau,\mu\in G} A_\tau \otimes A_\mu\right) \cap \bigoplus_{\tau\mu=\sigma} C_\tau \otimes C_\mu = \bigoplus_{\tau\mu=\sigma} A_\tau \otimes A_\mu. \qquad \Box$$

Proposition 3.6. *Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a graded coalgebra and let R be the graded associated ring. The following statements hold:*

(1) *If I is a two-sided graded ideal of R, then $I^\perp = \{x \in C \mid f(x) = 0\ \forall f \in I\}$ is a graded subcoalgebra of C.*

(2) *Let $J \subseteq R$ be a graded left ideal, then $J^\perp$ coincides with $Ann_C J = \{c \in C \mid Jc = 0\}$.*

(3) *Let $X \subseteq C$ be a left coideal and let $X^{\perp(R)} = \{f \in R | f(X) = 0\}$. Then, $X^{\perp(R)} = Ann_R(X) = \{f \in R \mid f \cdot x = 0;\ \forall x \in X\}$.*

(4) *Let $\rho_M : M \to M \otimes C$ be a graded right C-comodule and I be a graded two-sided ideal of R such that $I\overline{M} = 0$, then $\rho_M(M) \subseteq M \otimes I^\perp$ (i.e., M is a right graded comodule over the graded subcoalgebra $I^\perp$ of C).*

(5) *Let $A \subseteq C$ be a graded subcoalgebra of C. Then $A^{\perp(R)}$ is a graded two-sided ideal of R.*

(6) *If A is a graded subcoalgebra of C, then $(A^{\perp(R)})^\perp = A$.*

Proof. (1) is an immediate consequence of Lemma 3.5. (2)–(4) follow easily from [8, Proposition 4.1] (by taking in the proof homogeneous elements).

(5) Let A be a graded subcoalgebra of C. The inclusion $A \xrightarrow{i} C$ is graded morphism of coalgebras which induces a k-map $C^* \xrightarrow{i^*} A^*$, $(i^*(f) = f \circ i)$. Then, $i^*(R) \subseteq S$, where $S = \bigoplus_{\sigma \in G} S_\sigma$ is a G-graded ring, $S_\sigma = \{f \in A^* \mid f(A_\tau) = 0 \ \forall \tau \neq \sigma\}$. The kernel of $i^*|_R$ is precisely $A^{\perp(R)}$. Therefore, $A^{\perp(R)}$ is a graded two-sided ideal of R.

(6) The proof is an easy consequence of the comment given at the end of Section 2. □

Remark. (1) If C is a G-graded coalgebra, then C is, in particular, a right graded C-comodule. Exactly as in [3, Corollary 1], it results that ${}_RC$, as graded left R-module, is injective in $Rat(R-gr)$ (see [7] for the definition of $Rat(R-gr)$). Moreover, $\{C(\sigma)\}_{\sigma \in G}$ is a family of injective cogenerators in $Rat(R-gr)$ and $\bigoplus_{\sigma \in G} C(\sigma)$ is an injective cogenerator in $Rat(R-gr)$. In fact $Rat(R-gr) = \sigma_{R-gr}[\bigoplus_{\sigma \in G} C(\sigma)]$ (cf. [7]).

(2) Let W be a right graded C-comodule. For every k-space X, $X \otimes W$ is a right graded C-comodule with structure map

$$1_X \otimes \rho_W : X \otimes W \to X \otimes W \otimes C$$

which we denote, as in [3], by $(X) \otimes W$. It is clear that $(X) \otimes W$ is a direct sum of copies of W. Now, we see that $(X) \otimes W$ is "graded": $X \otimes W = X \otimes (\bigoplus_{\sigma \in G} W_\sigma) = \bigoplus_{\sigma \in G}(X \otimes W_\sigma)$; $1_X \otimes \rho_W(X \otimes W_\sigma) = X \otimes \rho_W(W_\sigma) \subseteq X \otimes (\sum_{\lambda\mu=\sigma} W_\lambda \otimes C_\mu) \subseteq \sum_{\lambda\mu=\sigma}(X \otimes W_\lambda) \otimes C_\mu = \sum_{\lambda\mu=\sigma}((X) \otimes W)_\lambda \otimes C_\mu$. So, $1_X \otimes \rho_W$ has the desired property.

Theorem 3.7. *Let C be a G-graded coalgebra and A be a graded subcoalgebra of C. We write $\mathscr{C}_A = \{M \in gr^C \mid \rho_M(M) \subseteq M \otimes A\}$. Then,*

(i) *$M \in \mathscr{C}_A \Leftrightarrow A^{\perp(R)}M = 0$.*

(ii) *$\mathscr{C}_A$ is a rigid closed subcategory of gr^C.*

(iii) *The map $A \mapsto \mathscr{C}_A$ is a bijective correspondence between the set of all graded subcoalgebras of C and the set of all rigid closed subcategories of gr^C.*

Proof. (i) Assume that $M \in \mathscr{C}_A$ and $f \in A^{\perp(R)}$. Then, $f(A) = 0$. Since $\rho_M(M) \subseteq M \otimes A$, then $0 = f \cdot m = \sum_{(m)} m_{(0)} f(m_{(1)})$ for any $m \in M$. Hence, $A^{\perp(R)}M = 0$. Conversely, if $A^{\perp(R)}M = 0$, by Proposition 3.6, $\rho_M(M) \subseteq M \otimes (A^{\perp(R)})^\perp = M \otimes A$.

(ii) From statement (i), it follows easily that $\mathscr{C}_A$ is closed under subobjects, quotients objects and direct sums. Also, for $M \in \mathscr{C}_A$, $\rho_{M(\sigma)}(M(\sigma)) = \rho_M(M) \subseteq M \otimes A = M(\sigma) \otimes A$. Hence, it is clear that $\mathscr{C}_A$ is rigid.

(iii) Let $\mathscr{C}$ be a rigid closed subcategory of gr^C. Since gr^C is isomorphic to $Rat(R-gr)$ (cf. [7, Theorem 6.3]), we can consider $\mathscr{C}$ as a rigid closed subcategory of $Rat(R-gr)$. But $Rat(R-gr)$ is a rigid closed subcategory of R-gr, [7, p. 476], so $\mathscr{C}$ can be viewed as a rigid closed subcategory of R-gr.

Now, we consider $\overline{A} = t_{\mathscr{C}}({}_R\overline{C})$ where $t_{\mathscr{C}}$ is the preradical associated to the closed subcategory $\mathscr{C}$ of R-gr. Then $\overline{A}$ is a left graded R-submodule of ${}_R\overline{C}$ (rational by construction). Let $g \in R_\tau$, $\tau \in G$. Since $\overline{C}$ is a graded $R-R$-bimodule, the map $u : \overline{C} \to \overline{C}(\tau)$, $u(c) = c \cdot g$ is a graded morphism of left R-modules. Since $\mathscr{C}$ is closed under quotient objects, $u(\overline{A}) \subseteq \overline{A}(\tau) = t_{\mathscr{C}}(\overline{C}(\tau))$, hence $\overline{A}g \subseteq \overline{A}(\tau)$; i.e., if $c \in \overline{A}_\sigma$, $g \in R_\tau$, $c \cdot g \in \overline{A}(\tau)_\sigma = \overline{A}_{\sigma\tau}$. Hence, $\overline{A}$ is a graded $R-R$-subbimodule of ${}_RC_R$. If we consider $A = \overline{\overline{A}}$ as a comodule then, by the preceding, we have $\Delta(A) \subseteq A \otimes C$ and $\Delta(A) \subseteq C \otimes A$; i.e., $\Delta(A) \subseteq (A \otimes C) \cap (C \otimes A) = A \otimes A$. It is clear that A is a "graded" subcoalgebra of C. Also, it can be seen easily that A, considered as a right graded C-comodule, belongs to $\mathscr{C}$. Let $M \in \mathscr{C}$. Because $\bigoplus_{\sigma \in G} C(\sigma)$ is an injective cogenerator in gr^C, we have a graded monomorphism $0 \to M \to (\bigoplus_{\sigma \in G} C(\sigma))^{(I)}$ for some $I \neq \emptyset$. Since $t_{\mathscr{C}}$ is left exact, and commutes with direct sums, we obtain $0 \to M \to (\bigoplus_{\sigma \in G} A(\sigma))^{(I)}$ (it is clear that $t_{\mathscr{C}}(C(\sigma)) = t_{\mathscr{C}}(C)(\sigma))$, where A is considered as a right graded C-comodule. Then $A^{\perp(R)}(\bigoplus_{\sigma \in G} A(\sigma))^{(I)} = 0$ implies $A^{\perp(R)}M = 0$, and $M \in \mathscr{C}_A$ by statement (i). Therefore $\mathscr{C} \subseteq \mathscr{C}_A$.

Conversely if $M \in \mathscr{C}_A$, then $\rho_M(M) \subseteq M \otimes A$. Hence M is a right graded A-comodule. Hence, there exists a right graded comodule monomorphism $0 \to M \to (\bigoplus_{\sigma \in G} A(\sigma))^{(I)}$ for some $I \neq \emptyset$. Since $A(\sigma) \in \mathscr{C}$ ($\mathscr{C}$ is rigid), $M \in \mathscr{C}$.

If A is a graded subcoalgebra of C and we put $B = t_{\mathscr{C}_A}(C)$, then from $\Delta(A) \subseteq A \otimes A$ it results that $A \subseteq B$. On the other hand by (i) we have $A^{\perp(R)}B = 0$, so by Proposition 3.6 we have $B \subseteq (A^{\perp(R)})^{\perp} = A$. So $A = B$. Now, it is easy to show that there exists a bijective correspondence between the set of all graded subcoalgebras of C and the set of all rigid closed subcategories of gr^C given by $A \mapsto \mathscr{C}_A$. □

Corollary 3.8. *If $\mathscr{C}$ is a rigid closed subcategory of gr^C, then $\mathscr{C}$ is closed under direct products.*

Proof. The proof is similar to the proof of [8, Corollary 4.3] so we omit it. □

Let $M \in gr^C$. We denote $I = Ann(\overline{M})$ ($\overline{M}$ is considered as a left graded R-module). Since I is a graded two-sided ideal of R, then $K = I^{\perp}$ is a graded subcoalgebra of C. The following result characterizes the closed subcategory $\sigma[M]$ of gr^C.

Corollary 3.9. *With the above notation, $\sigma[M] = \mathscr{C}_K$.*

Proof. This follows by the same method as the proof of [8, Corollary 4.4]. □

Remark. If $\mathscr{C}$ is a rigid closed subcategory of gr^C, then $\mathscr{C} = \mathscr{C}_A$ for some graded subcoalgebra A of C by Theorem 3.7. In fact, we have $\mathscr{C} = \sigma[A]$ where A is considered here as a right graded C-comodule.

As in the non-graded case [8], we are going to characterize the rigid closed subcategories of gr^C which are closed under extensions (that is, rigid localizing subcategories in gr^C).

Let $A \subseteq C$ be a graded subcoalgebra of C. Recall, [11], the inductive definition of $\Pi^n A$ as follows:

$$\Pi^0 A = \{0\}, \quad \Pi^1 A = A \quad \text{and} \quad \Pi^n A = \Pi^{n-1} A \Pi A,$$

where for X, Y k-subspaces of C, $X\Pi Y = \Delta^{-1}(C \otimes Y + X \otimes C)$. Following this notation, $\Pi^2 A = \Delta^{-1}(C \otimes A + A \otimes C)$. It is easy to check that $\Pi^2 A = (A^{\perp(R)^2})^{\perp}$ which is a graded subcoalgebra of C.

We will say that A is coidempotent if $\Pi^2 A = A$.

Analogous to [8, Theorem 4.5], we have the following result.

Theorem 3.10. *Let C be a graded coalgebra and let A be a graded subcoalgebra of C. Then, the following statements hold*:

(i) *If $\Pi^2 A = A$ (i.e. A is co-idempotent), then $\mathscr{C}_A$ is a rigid localizing subcategory of gr^C.*

(ii) *The map $A \mapsto \mathscr{C}_A$ is a bijective map between the set of all co-idempotent graded subcoalgebras of C and the set of all rigid localizing subcategories of gr^C.*

Proof. (i) It is enough to show that $\mathscr{C}_A$ is closed under extensions. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in gr^C with M_1, $M_3 \in \mathscr{C}_A$. By Theorem 3.7, $A^{\perp(R)}M_1 = A^{\perp(R)}M_3 = 0$. Therefore, $(A^{\perp(R)})^2 M_2 = 0$. This means that $\rho_{M_2}(M_2) \subseteq M_2 \otimes (A^{\perp(R)^2})^{\perp}$. But $(A^{\perp(R)^2})^{\perp} = A$ and so $\rho_{M_2}(M_2) \subseteq M_2 \otimes A$; i.e., $M_2 \in \mathscr{C}_A$.

(ii) Let $\mathscr{C}$ be a rigid localizing subcategory of gr^C. By Theorem 3.6(iii), $\mathscr{C} = \mathscr{C}_A$ for some graded subcoalgebra A of C. By the definition of "wedge" product $A \subseteq \Pi^2 A$ and $\Pi^2 A$ is a subcoalgebra of C. Therefore,

$$\Delta(\Pi^2 A) \subseteq A \otimes (\Pi^2 A) + (\Pi^2 A) \otimes A\,.$$

Hence, $\rho(\Pi^2 A/A) \subseteq (\Pi^2 A/A) \otimes A$, where ρ is the quotient C-comodule structure map. As A, $\Pi^2 A/A \in \mathscr{C}_A$, $\Pi^2 A \in \mathscr{C}_A$. Since $\Pi^2 A$ is a subcoalgebra of C, $\Pi^2 A = A$. $\square$

4. Relative Dade's theorem

Now, we give a coalgebraic version of the divisorially Dade's theorem (see [2] for the divisorially Dade's theorem for graded rings).

Let D be a coalgebra and let A be a coidempotent subcoalgebra of D. We will write $\mathscr{T}$ as the localizing subcategory associated to A in $\mathscr{M}^D$ and as $Q_{\mathscr{T}}$ the localization functor associated to A. Let M, N be two right D-comodules. We will say that a homomorphism of right D-comodules $f : M \to N$ is a *quasi-isomorphism* (relative to $\mathscr{T}$), if the induced map $Q_{\mathscr{T}}(f) : Q_{\mathscr{T}}(M) \to Q_{\mathscr{T}}(N)$ is an isomorphism of right D-comodules.

Since the localization functor $Q_{\mathscr{T}}(-) : \mathscr{M}^D \to \mathscr{M}^D$ is k-linear, left exact and preserves direct sums ($\mathscr{M}^D$ is a locally Noetherian Grothendieck category), by [12, Proposition 2.1] there is a (D,D)-bicomodule P such that $Q_{\mathscr{T}}(-) \cong (- \,\square_D\, P)$ as functors. So if M is a (D,D)-bicomodule, then $Q_{\mathscr{T}}(M)$ is a (D,D)-bicomodule and the canonical map $l_M : M \to Q_{\mathscr{T}}(M)$ is a (D,D)-bicomodule map.

Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and $\mathscr{T}$ a localizing subcategory in $\mathscr{M}^{C_e}$. We can induce a rigid localizing subcategory in gr^C given by: $\mathscr{T}^g = \{X \in gr^C \mid$

$U(X)_{C_e} \in \mathscr{T}\}$ where $U(X)_{C_e}$ is X considered as right C_e-comodule by means of the structure map $X \xrightarrow{\rho_X} X \otimes C \xrightarrow{1_X \otimes \pi_e} X \otimes C_e$. It is easy to see that $\mathscr{T}^g = \{X \in gr^C \mid X_\sigma \in \mathscr{T}\ \forall \sigma \in G\}$.

We write $\mathscr{T} - \mathscr{M}^{C_e}$ as the full subcategory of $\mathscr{T}$-closed comodules of $\mathscr{M}^{C_e}$ and, analogously, we write $\mathscr{T}^g - gr^C$ as the full subcategory of $\mathscr{T}^g$-closed graded comodules of gr^C. We will consider the following commutative square of categories and functors:

$$\begin{tikzcd}
gr^C \arrow[r, shift left, "(-)_e"] \arrow[d, shift right, "a^g"'] & \mathscr{M}^{C_e} \arrow[l, shift left, "(-\square_{C_e} C)"] \arrow[d, shift right, "a"'] \\
\mathscr{T}^g - gr^C \arrow[r, shift left, "G"] \arrow[u, shift right, "i^g"'] & \mathscr{T} - \mathscr{M}^{C_e} \arrow[l, shift left, "F"] \arrow[u, shift right, "i"']
\end{tikzcd}$$

where i^g, i are the inclusion functors and a^g, a are the reflectors. F is the composition $F = a^g \circ (- \square_{C_e} C) \circ i$ and $G = a \circ (-)_e \circ i^g$. We will consider the canonical maps $u^C_{\sigma,\tau} : C_{\sigma\tau} \to C_\sigma \otimes C_\tau$. Note that $\mathrm{Im}(u^C_{\sigma,\tau}) \subseteq C_\sigma \square_{C_e} C_\tau$ for all $\sigma, \tau \in G$. With this notation, we have the next theorem.

Theorem 4.1. *Let $C = \bigoplus_{\sigma \in G} C_\sigma$ be a G-graded coalgebra and $\mathscr{T}$ a localizing subcategory in $\mathscr{M}^{C_e}$. The following conditions are equivalent:*

(i) *The functor $F = a^g \circ (- \square_{C_e} C) \circ i$ and $G = a \circ (-)_e \circ i^g$ establish an equivalence between the categories $\mathscr{T}^g - gr^C$ and $\mathscr{T} - \mathscr{M}^{C_e}$.*

(ii) *A right graded C-comodule $X \in gr^C$ is in $\mathscr{T}^g$ if and only if $X_e \in \mathscr{T}$.*

(iii) *For every $\sigma, \tau \in G$, Ker$(C_{\sigma\tau} \xrightarrow{u^C_{\sigma\tau}} C_\sigma \square_{C_e} C_\tau) \in \mathscr{T}$. (Here, $u^C_{\sigma\tau} = (\pi^C_\sigma \otimes \pi^C_\tau) \circ \Delta \circ i^C_{\sigma\tau}$.) (When this condition is given, we will say that C is a $\mathscr{T}$-divisorially graded coalgebra.)*

Proof. (i) $\Rightarrow$ (ii) We only have to prove the following statement: given $X \in gr^C$, if $X_e \in \mathscr{T}$, then $X \in \mathscr{T}^g$.

We consider the exact sequence in gr^C: $0 \to K \to X \to Q_{\mathscr{T}^g}(X) \to D \to 0$, where $K, D \in \mathscr{T}^g$ and $Q_{\mathscr{T}^g}(-) = i^g \circ a^g$ is the localization functor. Since $(-)_e$ is exact we have the exact sequence in $\mathscr{M}^{C_e}$, $0 \to K_e \to X_e \to (Q_{\mathscr{T}^g}(X))_e \to D_e \to 0$. By applying the reflector a, we obtain $a(X_e) \cong a(Q_{\mathscr{T}^g}(X)_e)$ because $D_e, K_e \in \mathscr{T}$. The hypothesis $X_e \in \mathscr{T}$ implies that $0 = a(X_e) \cong a(Q_{\mathscr{T}^g}(X)_e) = a((i^g \circ a^g(X))_e) = G(a^g(X))$. Therefore, $a^g(X) = 0$ (G is an equivalence), so $X \in \mathscr{T}^g$.

(ii) $\Rightarrow$ (iii) For all $M \in gr^C$, there exists a natural morphism $\alpha(M) : M \to M_e \square_{C_e} C$ (cf. [7, Section 4]) given by $\alpha(M) = (\pi^M_e \otimes 1_C) \circ \rho$. We consider, for $\tau \in G$, the morphism $\alpha(C(\tau)) : C(\tau) \to C_\tau \square_{C_e} C$. By applying $(-)_e$, we obtain the isomorphism $(\alpha(C(\tau))_e : C(\tau)_e = C_\tau \to (C_\tau \square_{C_e} C)_e = C_\tau \square_{C_e} C_e \cong C_\tau$. By hypothesis,

$Ker(\alpha(C(\tau))_\sigma \in \mathscr{T}$ for all $\sigma,\tau \in G$. By [7, Theorem 5.4], $\alpha(C(\sigma))_\tau = u^C_{\sigma,\tau}$ and so $Ker(u^C_{\sigma,\tau}) \in \mathscr{T}$ for all $\sigma,\tau \in G$.

(iii) $\Rightarrow$ (ii) Let $X \in gr^C$ and suppose that $X_e \in \mathscr{T}$. We consider the natural morphism $\alpha(X) : X \to X_e \square_{C_e} C$ and we apply $(-)_\sigma$, $\sigma \in G$: $\alpha(X)_\sigma : X_\sigma \to X_e \square_{C_e} C_\sigma$. We have the short exact sequence:

$$0 \to Ker(\alpha(X)_\sigma) \to X_\sigma \to \mathrm{Im}(\alpha(X)_\sigma) \to 0. \qquad (*).$$

Since $X_e \in \mathscr{T}$, $X_e \square_{C_e} C_\sigma \in \mathscr{T}$, by Lemma 3.2. Therefore, $\mathrm{Im}(\alpha(X)_\sigma) \in \mathscr{T}$. Now, we will see that $Ker(\alpha(X)_\sigma) \in \mathscr{T}$. We know that $\alpha(X)_\sigma = u^X_{e,\sigma}$. By [7, Proposition 3.1], we have the commutative diagram

$$\begin{tikzcd}
X_\sigma \arrow[r, "u^X_{\sigma,e}"] \arrow[d, "u^X_{\sigma\tau,\tau^{-1}}"'] & X_\sigma \square_{C_e} C_e \arrow[d, "1\square u_{\tau,\tau^{-1}}"] \\
X_{\sigma\tau} \square_{C_e} C_{\tau^{-1}} \arrow[r, "u^X_{\sigma,\tau}\square 1"'] & X_\sigma \square_{C_e} C_\tau \square_{C_e} C_{\tau^{-1}}
\end{tikzcd}$$

for all $\sigma,\tau \in G$.

By [7, Proposition 3.1], $u^X_{\sigma,e}$ is a monomorphism and, by hypothesis, $u^C_{\tau,\tau^{-1}}$ has kernel in $\mathscr{T}$. Hence, it is easy to see that the composition $(1 \square u^C_{\tau,\tau^{-1}}) \circ u^X_{\sigma,e}$ has kernel in $\mathscr{T}$. Therefore, by the commutativity of the diagram, $u^X_{\sigma\tau,\tau^{-1}}$ has kernel in $\mathscr{T}$. By taking $\tau = \sigma^{-1}$, it follows that $Ker(u^X_{e,\sigma}) = Ker(\alpha(X)_\sigma) \in \mathscr{T}$.

Since $\mathscr{T}$ is closed under extensions, we have (by considering the sequence $(*)$) that $X_\sigma \in \mathscr{T}$ for all $\sigma \in G$. So, $X \in \mathscr{T}^g$.

(ii) $\Rightarrow$ (i): First, we prove the following statement:

(*) “if $X \in gr^C$ is $\mathscr{T}^g$-closed, then X_e is $\mathscr{T}$-closed.

Let $P \xrightarrow{f} Q$ be a quasi-isomorphism in $\mathscr{M}^{C_e}$ relative to $\mathscr{T}$. Then

$$P \square_{C_e} C \xrightarrow{f\square_{C_e} C} Q \square_{C_e} C$$

is also a quasi-isomorphism in gr^C relative to $\mathscr{T}^g$ since $(- \square_{C_e} C)$ is exact. Therefore, $Hom_{gr^C}(P \square_{C_e} C,X) \cong Hom_{gr^C}(Q \square_{C_e} C,X)$ and hence

$$Hom_{C_e}(P,X_e) \overset{f^*}{\cong} Hom_{C_e}(Q,X_e).$$

Now, we prove statement (i). Take $M \in \mathscr{T} - \mathscr{M}^{C_e}$, then $GF(M) = a(Q_{\mathscr{T}^g}(i(M) \square_{C_e} C))_e$. Consider the exact sequence in gr^C,

$$0 \to K \to i(M) \square_{C_e} C \to Q_{\mathscr{T}^g}(i(M) \square_{C_e} C) \to D \to 0$$

with $K,D \in \mathscr{T}^g$. By applying the functor $(-)_e$ to the above exact sequence, we have: $0 \to K_e \to i(M) \to (Q_{\mathscr{T}^g}(i(M) \square_{C_e} C))_e \to D_e \to 0$. Since $i(M)$ and

$(Q_{\mathscr{T}^g}(i(M)\square_{C_e}C))_e$ are in $\mathscr{T}-\mathscr{M}^{C_e}$ and $K_e, D_e \in \mathscr{T}$, it follows that $K_e = D_e = 0$ and $i(M) \cong (Q_{\mathscr{T}^g}(i(M)\square_{C_e}C))_e$. Therefore, $GF(M) = a(Q_{\mathscr{T}^g}(i(M)\square_{C_e}C))_e \cong aiM \cong M$. Let $X \in \mathscr{T}^g - gr^C$, then $FG(X) = a^g(ia(i^gX)_e \square_{C_e} C) \cong a^g((i^gX)_e \square_{C_e} C)$. Since $\alpha(i^gX) : i^gX \to (i^gX)_e \square_{C_e} C$ has kernel and cokernel in $\mathscr{T}^g$, we deduce that $X \cong a^g(i^gX) \cong a^g((i^gX)_e \square_{C_e} C) \cong a^g(ia(i^gX)_e \square_{C_e} C) = FG(X)$. $\square$

Corollary 4.2. *If C is a $\mathscr{T}$-divisorially G-graded coalgebra and $\mathscr{T} \neq \mathscr{M}^{C_e}$, then the group G is finite.*

Proof. Let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a $\mathscr{T}$-divisorially G-graded coalgebra. For $x \in C_e$,

$$\Delta_C(x) \in \bigoplus_{\sigma\in S} C_\sigma \otimes C_{\sigma^{-1}}$$

for some finite subset S of G. Suppose G is infinite. Then there exists $\sigma_0 \in G - S$. Thus, $x \in \ker u^C_{\sigma_0,\sigma_0^{-1}}$. Therefore, $C_e \in \mathscr{T}$ and hence $\mathscr{T} = \mathscr{M}^{C_e}$, a contradiction. $\square$

Example. Let $C = C_0 \oplus C_1$ be a strongly $\mathbb{Z}_2$-graded coalgebra ($\mathscr{M}_2^c(k)$, the matrix coalgebra over a field k, is an easy example of such coalgebra). Let B be any group-like coalgebra. Then $D = (C_0 \oplus B) \oplus C_1$, with the obvious comultiplication and counit, is a $\mathbb{Z}_2$-graded coalgebra, $C_0 \oplus B$ being the degree zero component and C_1 being the degree one component. It is easy to see that B is a coidempotent subcoalgebra of $C_0 \oplus B$ and D is a $\mathscr{C}_B$-divisorially graded coalgebra.

5. Divisorially graded coalgebras and relative Picard groups

Throughout this section, we write $C = \bigoplus_{g\in G} C_g$ as a G-graded coalgebra. The aim of this section is to give a structure theorem of divisorially graded coalgebras in terms of the Picard group for a coalgebra respect to a localizing subcategory.

The construction of the (non relative) Picard group for a coalgebra was introduced in [13]. Relative Picard group for algebras can be consulted in [14].

We will inspire in [14, p. 49] to make the following definition.

Definition 5.1. Let D be a coalgebra and let A be a coidempotent subcoalgebra of D. Let $\mathscr{T}$ be the localizing subcategory associated to A in $\mathscr{M}^D$.

A (D,D)-bicomodule N will be called $\mathscr{T}$-coinvertible in case that

(i) $T \square_D N \in \mathscr{T}$ for all $T \in \mathscr{T}$,

(ii) if $X \to Y$ is an epimorphism in $\mathscr{M}^D$, then

$$Coker(X \square_D N \to Y \square_D N) \in \mathscr{T},$$

and

(iii) there exists a (D,D)-bicomodule L and isomorphisms in ${}^D\mathscr{M}^D$:

$$Q_{\mathscr{T}}(N \square_D L) \cong Q_{\mathscr{T}}(D), \quad Q_{\mathscr{T}}(L \square_D N) \cong Q_{\mathscr{T}}(D).$$

Definition 5.2. Let D be a coalgebra and let A be a coidempotent subcoalgebra of D. Define in ${}^{D}\mathscr{M}^{D}$ the following equivalence relation:

$$M \equiv N \;\Leftrightarrow\; Q_{\mathscr{T}}(M) \cong Q_{\mathscr{T}}(N).$$

Then, the Picard group of D relative to A, denoted by $Pic(D,A)$, is the multiplicative group consisting of all (D,D)-bicomodule isomorphism classes $[N]$ of $\mathscr{T}$-coinvertible bicomodules. Multiplication is given by

$$[N][L] = [N \,\square_D\, L].$$

The class $[D]$ is the identity element of $Pic(D,A)$.

We need to develop some tools in order to prove that $Pic(D,A)$ is well defined.

Lemma 5.3. *In the conditions above, let N, L be two bicomodules. Then:*
(a) *If L verifies* (i) *and* (ii) *in the definition of $\mathscr{T}$-coinvertible bicomodule, then $Q_{\mathscr{T}}(Q_{\mathscr{T}}(N) \,\square_D\, L) \cong Q_{\mathscr{T}}(N \,\square_D\, L)$,*
(b) *$Q_{\mathscr{T}}(N \,\square_D\, Q_{\mathscr{T}}(L)) \cong Q_{\mathscr{T}}(N \,\square_D\, L)$.*

Proof. (a) Let

$$0 \to t(N) \to N \to Q_{\mathscr{T}}(N) \to S \to 0$$

be exact in $\mathscr{M}^{D}$, where $N \to Q_{\mathscr{T}}(N)$ is the canonical map and $t(N), S \in \mathscr{T}$. This sequence splits into two short exact sequences:

$$0 \to t(N) \to N \to X \to 0$$

and

$$0 \to X \to Q_{\mathscr{T}}(N) \to S \to 0.$$

By hypothesis the sequence

$$0 \to t(N) \,\square_D\, L \to N \,\square_D\, L \to X \,\square_D\, L \to W \to 0$$

has $t(N) \,\square_D\, L$, $W \in \mathscr{T}$. So, $Q_{\mathscr{T}}(N \,\square_D\, L) \cong Q_{\mathscr{T}}(X \,\square_D\, L)$. On the other hand, similar argument with the other short exact sequence gives

$$Q_{\mathscr{T}}(X \,\square_D\, L) \cong Q_{\mathscr{T}}(Q_{\mathscr{T}}(N) \,\square_D\, L)$$

and we obtain the thesis.

(b) Let $Q_{\mathscr{T}}(-) = (- \,\square_D\, P)$ for some $P \in {}^{D}\mathscr{M}^{D}$. Since $Q_{\mathscr{T}}(Q_{\mathscr{T}}(D)) \cong Q_{\mathscr{T}}(D)$ as (D,D)-bicomodules, $P \,\square_D\, P \cong P$ as (D,D)-bicomodules. Therefore,

$$Q_{\mathscr{T}}(N \,\square_D\, Q_{\mathscr{T}}(L)) \cong N \,\square_D\, L \,\square_D\, P \,\square_D\, P \cong N \,\square_D\, L \,\square_D\, P \cong Q_{\mathscr{T}}(N \,\square_D\, L)$$

as (D,D)-bicomodules. $\square$

Corollary 5.4. *$Pic(D,A)$ is a well-defined group with the multiplication given above.*

Proof. It is enough to check that the multiplication does not depend on the representatives choosen and that if N, L are $\mathscr{T}$-coinvertible, then $N \square_D L$ is too.

Suppose $[N] = [N']$ and $[L] = [L']$. Then, by using Lemma 5.3, we have $N \square_D L \equiv Q_{\mathscr{T}}(N) \square_D L \equiv Q_{\mathscr{T}}(N') \square_D L \equiv N' \square_D L \equiv N' \square_D Q_{\mathscr{T}}(L) \equiv N' \square_D Q_{\mathscr{T}}(L') \equiv N' \square_D L'$. So, $[N \square_D L] = [N' \square_D L']$.

On the other hand, let L, N be $\mathscr{T}$-coinvertible bicomodules and let L', N' be such that $Q_{\mathscr{T}}(N \square_D N') \cong Q_{\mathscr{T}}(N' \square_D N) \cong Q_{\mathscr{T}}(D)$ and similar property for L' respect to L. It is easy to see that $N \square_D L$ verifies (i) and (ii) in the definition of $\mathscr{T}$-coinvertible bicomodule.

For (iii), we have $L' \square_D N' \square_D N \square_D L \equiv Q_{\mathscr{T}}(L' \square_D N' \square_D N) \square_D L \equiv Q_{\mathscr{T}}(L' \square_D Q_{\mathscr{T}}(N' \square_D N)) \square_D L \equiv Q_{\mathscr{T}}(L') \square_D L \equiv L' \square_D L \equiv D$; and analogously $N \square_D L \square_D L' \square_D N' \equiv D$. So (iii) is verified. $\square$

Suppose that C is a $\mathscr{T}$-divisorially G-graded coalgebra. Note that, in this case, by using Theorem 4.1, the map $\varphi : G \to Pic(C_e, A)$ given by $\varphi(\sigma) = [C_\sigma]$ for all $\sigma \in G$, is a group homomorphism (C_σ is $\mathscr{T}$-coinvertible for all $\sigma \in G$ since, by Lemma 3.2, $\mathscr{T}$ is G-invariant (condition (i) of $\mathscr{T}$-coinvertible bicomodule) and conditions (ii) and (iii) are given by Theorem 4.1).

The next definition is, in some sense, dual to the definition of factor set given in [5].

Definition 5.5. Let D be a coalgebra and let A be a coidempotent subcoalgebra of D. Let G be a finite group. Let $\phi : G \to Pic(D, A)$ be a group homomorphism. We write $\phi(\sigma) = [N_\sigma]$.

A cofactor set $\mathscr{F}$ associated to ϕ is a set $\mathscr{F} = \{f_{\tau,\sigma} \mid \tau, \sigma \in G\}$ of (D, D)-bicomodules quasi-isomorphisms: $f_{\tau,\sigma} : N_{\tau\sigma} \to N_\tau \square_D N_\sigma$, $\omega : N_e \to D$, such that the following diagrams commute, for all $\mu, \lambda, \tau \in G$:

$$\begin{tikzcd} N_{\mu\lambda\tau} \arrow[r, "f_{\mu\lambda,\tau}"] \arrow[d, "f_{\mu,\lambda\tau}"] & N_{\mu\lambda} \square N_\tau \arrow[d, "f_{\mu,\lambda} \square 1_\tau"] \\ N_\mu \square N_{\lambda\tau} \arrow[r, "1_\mu \square f_{\lambda,\tau}"] & N_\mu \square N_\lambda \square N_\tau \end{tikzcd} \qquad \begin{tikzcd} N_\lambda \square N_e \arrow[rr, "1 \square \omega"] & & N_\lambda \square D \\ & N_\lambda \arrow[ul, "f_{\lambda,e}"] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} N_e \square N_\lambda \arrow[rr, "\omega \square 1"] & & D \square N_\lambda \\ & N_\lambda \arrow[ul, "f_{e,\lambda}"] \arrow[ur] & \end{tikzcd}$$

We write $\mathbf{F}_s(\phi)$ for the set of cofactor sets associated to ϕ. We make the k-space $\bigoplus_{\sigma\in G} N_\sigma$ into a coalgebra by defining comultiplication and counit as follows: for $n \in N_\sigma$ define $\Delta(n) = \sum_{\lambda\mu=\sigma} f_{\lambda,\mu}(n)$ and given $x \in N_e$, $\varepsilon(x) = \varepsilon_D \circ \omega(x)$, $\varepsilon(N_\sigma) = \{0\}$ for all $\sigma \neq e$. This coalgebra will be written $D\langle \mathscr{F}, \phi, G\rangle$.

Theorem 5.6. *The coalgebra $D\langle \mathscr{F}, \phi, G\rangle$ is a $\mathscr{T}$-divisorially graded coalgebra with coalgebra of degree e quasi-isomorphic to D, i.e., $Q_{\mathscr{T}}(N_e) \cong Q_{\mathscr{T}}(D)$. Conversely, if C is a $\mathscr{T}$-divisorially G-graded coalgebra, then there exists a group homomorphism $\phi : G \to Pic(C_e, A)$ and a cofactor set $\mathscr{F} \in \mathbf{F}_s(\phi)$ such that C is isomorphic to $C_e\langle \mathscr{F}, \phi, G\rangle$.*

Proof. First, we show that $(D\langle \mathscr{F}, \phi, G\rangle, \Delta, \varepsilon)$ defined above is a coalgebra. We take $\tau \in G$ and $x \in N_\tau$. For $\lambda, \mu \in G$ with $\lambda\mu = \tau$ we write $f_{\lambda,\mu}(x) = \sum x_\lambda \otimes x_\mu$ and $f_{s,t}(x_\mu) = \sum x_{\mu,s} \otimes x_{\mu,t}$, $(st = \mu)$. Then, the equalities $(f_{a,b} \otimes 1_c) f_{ab,c} = (1_a \otimes f_{b,c}) f_{a,bc}$ $(a, b, c \in G)$ imply that $\sum (x_{ab,a} \otimes x_{ab,c} \otimes x_c) = \sum (x_a \otimes x_{bc,b} \otimes x_{bc,c})$. With this notation, we have $(1 \otimes \Delta)\Delta(x) = \sum_{\lambda\mu=\tau} \sum (x_\lambda \otimes \Delta(x_\mu)) = \sum_{\lambda\mu=\tau} \sum \sum_{st=\mu} \sum (x_\lambda \otimes x_{\mu,s} \otimes x_{\mu,t}) = \sum (x_\lambda \otimes x_{st,s} \otimes x_{st,t}) = \sum (x_{\lambda s,\lambda} \otimes x_{\lambda s,s} \otimes x_t) = \sum (\sum (x_{\lambda s,\lambda} \otimes x_{\lambda s,s}) \otimes x_t) = \sum (\Delta(x_{\lambda s}) \otimes x_t) = (\Delta \otimes 1)(\sum x_{\lambda s} \otimes x_t) = (\Delta \otimes 1)\Delta(x_{\lambda st})$, where $x = x_{\lambda st}$. So, $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$. On the other hand, $(1 \otimes \varepsilon)\Delta(x) = (1 \otimes \varepsilon)(\sum_{\lambda\mu=\tau} f_{\lambda,\mu})(x) = (1 \otimes \varepsilon) f_{\tau,e}(x) = (1 \otimes \varepsilon_D)(1 \otimes \omega) f_{\tau,e}(x) = (1 \otimes \varepsilon_D)\rho_\tau^+(x) = x$, where $\rho_\tau^+ : N_\tau \to N_\tau \,\square\, D$ is the structure map of N_τ as right D-comodule. So, $(1 \otimes \varepsilon)\Delta = 1$. Analogously $(\varepsilon \otimes 1)\Delta = 1$.

It is clear, by the definition of cofactor set, that $D\langle \mathscr{F}, \phi, G\rangle$ is a divisorially G-graded coalgebra. Now, we show that the (D, D)-bicomodule quasi-isomorphism, $\omega : N_e \to D$ is also a coalgebra map: $(\omega \otimes \omega)\Delta_e = (1 \otimes \omega)(\omega \otimes 1) f_{e,e} = (1 \otimes \omega)\rho_e^- = \Delta_D \omega$. Therefore $(\omega \otimes \omega)\Delta_e = \Delta_D \omega$. Finally, $\varepsilon_D \omega = \varepsilon$ by definition. Hence $w : N_e \to D$ is a coalgebra map.

For the converse of the theorem, let $C = \bigoplus_{\sigma\in G} C_\sigma$ be a divisorially graded coalgebra. It is clear, by using [7, Proposition 3.1], that the map $\phi : G \to Pic(C_e, A)$ given by $\phi(\tau) = [C_\tau]$ and the set $\{u_{\sigma,\tau} : C_{\sigma\tau} \to C_\sigma \,\square\, C_\tau,\ \ \sigma, \tau \in G\}$ is the group homomorphism and the cofactor set desired, respectively. $\square$

Remark. It is easy to see that a strongly graded coalgebra is the particular case of $\mathscr{T}$-divisorially graded coalgebra when $\mathscr{T} = \{0\}$. In this last case, theorem above can be reformulated to get the non-relative analogue.

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